

ANALYTICAL CLASSIFICATION OF BUCKLING PROBLEMS OF COMPRESSED RODS WITH VARIABLE BENDING STIFFNESS USING SPECIAL FUNCTIONS

Mirosław SADOWSKI*

Faculty of Engineering and Technical Sciences, University of Zielona Góra, Zielona Góra, Poland

A b s t r a c t

The paper presents an analytical framework for the classification of buckling problems of compressed rods with variable bending stiffness. The governing Euler–Bernoulli stability equation with spatially varying coefficients is transformed into a Sturm–Liouville form and further reduced to a Schrödinger-type equation using the Liouville transformation. This formulation establishes a direct correspondence between the bending stiffness distribution and the associated spectral problem, allowing a systematic mapping of stiffness profiles to classes of differential equations. Depending on the resulting Liouville potential, the eigenvalue problems can be expressed in terms of classical special functions such as Airy, Bessel, Hermite, or error-function-based solutions. The proposed approach provides a unified classification of variable-stiffness buckling problems within a single operator framework and organizes known analytical results as special cases of a general spectral formulation. The study focuses on analytical structure and classification rather than on optimization procedures, and it does not assume or solve a shape optimization problem.

Keywords: elastic stability, variable bending stiffness, special functions, Euler–Lagrange equation, Sturm–Liouville equation

1. INTRODUCTION

The origins of modern elastic stability theory are associated with the work of Euler [1], in which a model of an axially compressed rod was formulated and the critical force was derived, and with Lagrange [2], who introduced variational methods into stability analysis. The development of metal structures in the eighteenth and nineteenth centuries (Timoshenko [3]) revealed the importance of slender compressed structural elements and established stability as a key engineering problem. In the nineteenth and early twentieth centuries, both experimental and theoretical foundations of stability developed rapidly, with contributions by Fairbairn [4], Bauschinger [5], Jasiński [6], Tetmajer [7], Bresse [8], Thompson [9], Lapunov [10], and Bryan [11], who formed the early basis of elastic stability theory.

* Corresponding author: Mirosław Sadowski, miroslaw.sadowski@gmail.com, Independent Researcher, Zgorzelec, Poland.

Further development was driven by Lagrange [12] and Clausen [13], who introduced variational and geometric aspects of optimal structural forms. Subsequent works by Keller et al. [14], Keller [15], Gajewski [16,17], and Filipov et al. [18–20] extended these ideas to more general materials, geometries, and loading conditions. Approximate and numerical approaches were developed by Krzyś et al. [21], Błachut [22], Olhoff et al. [23], and Marcinowski [24], showing that optimized profiles can significantly increase buckling resistance.

A modern treatment of stability is presented in Timoshenko et al. [25], while later studies emphasize that real structures often exhibit variable stiffness along their length. This leads to stability equations with variable coefficients and motivates research on non-uniform rods and beams, including works on variable volume profiles (Lee et al. [26]), nonuniform cross-sections (Konstantakopoulos et al. [27]), and stepped rods. In parallel, shape optimization of compressed rods has been formulated as an eigenvalue problem, where optimal stiffness distributions maximize the critical load. Numerical and semi-analytical methods remain dominant (Konstantakopoulos et al. [27], Nikolić et al. [29], Błażejowski et al. [30]). Analytical approaches based on special functions and closed-form solutions were proposed by Marcinowski et al. [31,32], with reported increases in buckling capacity [33]. More recent works (Dokšanović et al. [34], Turkyilmazoglu [35], Di Re [36]) confirm that most variable-stiffness problems are still treated using numerical or semi-analytical methods. The main research gap is the lack of a unified framework showing that different stiffness distributions lead to a common Sturm–Liouville structure with solutions expressed in terms of special functions determined by the corresponding potential.

In summary, there exists a rich body of literature concerning the buckling of rods with variable bending stiffness, shape optimization, and the analysis of functionally graded material (FGM) elements. However, most works treat specific forms of the cross-sectional moment of inertia as separate problems solved by different methods. There is a lack of a coherent mathematical theory showing that they lead to the same class of eigenvalue equations (known in the literature as Sturm – Liouville equations), and that solutions appearing, for example, in terms of Airy, Bessel, Hermite, or erf/erfi-based functions are different realizations of a single equation in normalized form.

The aim of the present paper is to propose such a unified analytical framework. By means of the Liouville transformation, the buckling equation of a rod with variable bending stiffness can be reduced to the form

$$\phi''(s) + (\lambda - V(s))\phi(s) = 0, \quad (1.1)$$

where the potential $V(s)$ uniquely corresponds to the profile of the cross-sectional moment of inertia.

This makes it possible to construct a kind of “mathematical dictionary” linking selected classes of cross-sectional moment of inertia profiles with their corresponding potentials and special functions, which constitutes a foundation for symbolic analysis and analytical spectral classification based on the investigation of the first eigenvalues of the spectrum of the Sturm – Liouville operator.

2. PROBLEM FORMULATION

In the subsequent analyses, an axially compressed rod, pinned at both ends, is considered. Its static scheme is shown in Fig. 1. The loading system consists of a force F acting along the axis of the element.

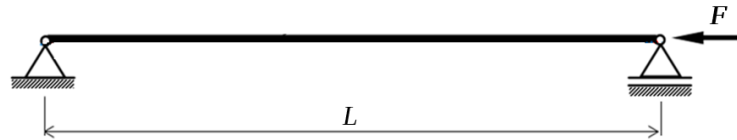


Fig. 1. Static scheme of the system

Due to the fact that the rod is pinned at both ends, the sought deflection function satisfies the boundary conditions

$$[w(x)]_{0,L} = [w''(x)]_{0,L} = 0. \quad (2.1)$$

These conditions determine the admissible class of functions for the eigenvalue problem resulting from the stability equation, while the buckling mode shape itself is not assumed *a priori* but follows from the solution of the differential equation in the subsequent part of the study.

2.1. Variable bending stiffness

It is assumed that the second moment of area of the cross-section varies along the length of the rod according to a smooth function $f(x)$ of class C^2

$$EI(x) = EI_0 f(x), \quad (2.2)$$

where I_0 is the value of the second moment of area defined at the mid-length of the rod, $f(x)$ is a positive, smooth profile shape function (Fig. 2), and E is Young's modulus describing the material properties. A change in the function $f(x)$ therefore corresponds to a change in the geometry and the local bending stiffness of the rod, which directly affects its buckling load capacity.

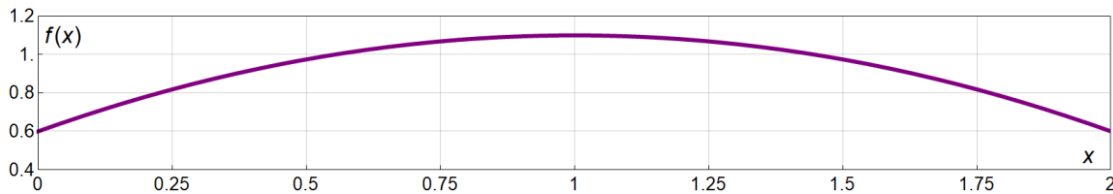


Fig. 2. Example of a function $f(x)$ describing the variation of bending stiffness

The function $f(x)$ represents a normalized distribution of bending stiffness along the rod axis and corresponds to continuous variation of stiffness.

2.2. Stability equation of a rod with variable bending stiffness

2.2.1. Total potential energy of the system

The total potential energy $\Pi[w]$ is the sum of the elastic bending strain energy and the potential energy of the axial force (the work of the compressive force). The bending moment is given by (Love [37])

$$M(x) = EI(x)\kappa(x), \quad (2.3)$$

where (Romanów [38])

$$\kappa(x) = \frac{w''(x)}{(1 + (w'(x))^2)^{3/2}} \quad (2.4)$$

denotes the curvature of the rod axis. It is assumed that the rod deflections are infinitesimal, and therefore $w'(x) \approx 0$, which leads to the relation (Landau et al. [39])

$$\kappa(x) = w''(x). \quad (2.5)$$

In view of the above, the elementary bending strain energy takes the form (Sadowski [40])

$$dU = \frac{M^2(x)}{2EI(x)} dx = \frac{1}{2}EI(x)(w''(x))^2 dx, \quad (2.6)$$

whereas the total elastic energy is given by

$$U[w] = \frac{1}{2} \int_L EI(x)(w''(x))^2 dx. \quad (2.7)$$

The force F acts axially, and its work (in the second-order approximation) during the transition to the buckled state is related to the change in the length of the rod axis; for small deflections (Sadowski [39])

$$ds = \sqrt{1 + (w'(x))^2} dx \approx \left(1 + \frac{1}{2}(w'(x))^2\right) dx, \quad (2.8)$$

hence

$$\Delta s = ds - dx = \frac{1}{2}(w'(x))^2 dx. \quad (2.9)$$

The elementary work of the compressive force is expressed by

$$dW = -F\Delta s = -\frac{1}{2}F(w'(x))^2 dx, \quad (2.10)$$

whereas the potential energy of the load is (Volmir [41])

$$W[w] = -\frac{1}{2}F \int_L (w'(x))^2 dx. \quad (2.11)$$

The total potential energy is given by the relation

$$\Pi[w] = U[w] + W[w] = \frac{1}{2} \int_L EI(x)(w''(x))^2 dx - \frac{1}{2}F \int_L (w'(x))^2 dx. \quad (2.12)$$

2.2.2. Euler – Lagrange equation (stability equation)

The equilibrium condition in the critical state follows from the stationarity of the functional $\Pi[w]$, which is satisfied by the condition $\delta\Pi[w] = 0$. The first variation is

$$\delta\Pi = \int_L EI(x)w''(x)\delta w''(x)dx - F \int_L w'(x)\delta w'(x)dx. \quad (2.13)$$

In the subsequent transformations, the first term of relation (2.13) yields

$$\int_L EI(x)w''(x)\delta w''(x)dx = \underbrace{[EI(x)w''(x)\delta w'(x)]_{0,L}}_{=0} - \int_L (EI(x)w''(x))'\delta w'(x)dx =$$

$$\begin{aligned}
 &= - \underbrace{\left[(EI(x) w''(x))' \delta w(x) \right]_{0,L}}_{=0} + \int_L (EI(x) w''(x))'' \delta w(x) dx = \\
 &= \int_L (EI(x) w''(x))'' \delta w(x) dx, \tag{2.14}
 \end{aligned}$$

whereas the second term of relation (2.13) gives

$$\begin{aligned}
 F \int_L w'(x) \delta w'(x) dx &= F \underbrace{[w'(x) \delta w(x)]_{0,L}}_{=0} - F \int_L w''(x) \delta w(x) dx = \\
 &= -F \int_L w''(x) \delta w(x) dx \tag{2.15}
 \end{aligned}$$

Finally, the first variation takes the form

$$\delta \Pi = \int_L ((EI(x) w''(x))'' + F w''(x)) \delta w(x) dx. \tag{2.16}$$

The condition $\delta \Pi = 0$ leads to the so-called Euler – Lagrange equation (Timoshenko et al. [3])

$$(EI(x) w''(x))'' + F w''(x) = 0. \tag{2.17}$$

This is the stability equation of an Euler – Bernoulli rod with variable bending stiffness $EI(x)$ under axial compressive force F .

2.3. Stability equation in normalized form

2.3.1. Stability equation in Sturm – Liouville form

The stability equation (2.17) is a fourth-order differential equation with variable coefficients. By the classical substitution

$$u(x) = w''(x), \tag{2.18}$$

it can be reduced to an equivalent form, namely to a second-order equation with variable coefficients

$$(EI(x) u(x))'' + F u(x) = 0. \tag{2.19}$$

Introducing the auxiliary variable $u(x) = w''(x)$ does not impose additional boundary conditions. The conditions on u follow directly from the pinned–pinned support conditions of the original problem and are preserved through the transformation. Since the auxiliary variable is defined in terms of curvature, the transformation ensures equivalence of the boundary-value structure between the original Euler–Bernoulli formulation and the reduced Sturm–Liouville problem.

For the pinned–pinned rod, the original boundary conditions imply that the bending moment vanishes at both ends, which in the Euler–Bernoulli formulation is equivalent to $w''(0) = w''(L) = 0$. Since the auxiliary variable is defined as $u(x) = w''(x)$, the transformed problem satisfies homogeneous boundary conditions $u(0) = u(L) = 0$. Consequently, the reduced second-order equation preserves the boundary-value structure of the original fourth-order problem and constitutes a Sturm–Liouville eigenvalue problem. The eigenvalue F remains directly associated with the critical buckling load.

Relation (2.19) can be written as follows

$$EI(x)u''(x) + 2(EI(x))'u'(x) + ((EI(x))'' + F)u(x) = 0, \quad (2.20)$$

or equivalently

$$u''(x) + 2\frac{(EI(x))'}{EI(x)}u'(x) + \left(\frac{(EI(x))''}{EI(x)} + \frac{F}{EI(x)}\right)u(x) = 0. \quad (2.21)$$

We now introduce the substitution

$$u(x) = \psi(x) \exp\left(-\int^x \frac{(EI(\xi))'}{EI(\xi)} d\xi\right). \quad (2.22)$$

For the ratio $(EI(x))'/EI(x)$ the equality holds

$$\frac{(EI(x))'}{EI(x)} = \frac{d}{dx} \ln(EI(x)), \quad (2.23)$$

and its integration leads to

$$\int^x \frac{(EI(\xi))'}{EI(\xi)} d\xi = \ln(EI(x)) + c \Rightarrow \exp\left(-\int^x \frac{(EI(\xi))'}{EI(\xi)} d\xi\right) = \frac{c}{EI(x)},$$

where $c = \text{const}$, which transforms substitution (2.22) into the form

$$u(x) = c \frac{\psi(x)}{EI(x)}. \quad (2.24)$$

Taking (2.24) into account in equation (2.21) allows it to be written in the form

$$\psi''(x) + \frac{F}{EI(x)}\psi(x) = 0, \quad (2.25)$$

or

$$-\psi''(x) = \frac{F}{EI(x)}\psi(x), \quad (2.26)$$

which is consistent with the classical form of the *Sturm–Liouville differential equation* (Coddington et al. [42])

$$-(pf')' + qf = \lambda rf, \quad (2.27)$$

with $p = 1$, $q = 0$, $r = 1/(EI(x))$, $\lambda = F$.

The transformation preserves the physical structure of the original Euler–Bernoulli boundary-value problem. The pinned–pinned boundary conditions on the transverse displacement are satisfied in the transformed formulation through the definition of the auxiliary curvature-related variable. As a result, the Sturm–Liouville problem remains mathematically and physically equivalent to the original fourth-order stability problem. The eigenfunction represents a curvature-related state variable, while the eigenvalue is directly related to the critical buckling load.

2.3.2. Stability equation in normalized form

According to the theory of differential equations, in order to transform relation (2.27) into normalized form, the so-called Liouville transformation is applied (Ince [43], Titchmarsh [44]):

$$s = s(x) = \int_0^x \sqrt{r(t)/p(t)} dt, \quad (2.28)$$

$$\phi(s) = \phi(s(x)) = [p(x)r(x)]^{1/4} \psi(x),$$

which, in the case of equation (2.26), implies

$$s = s(x) = \int_0^x \frac{dt}{\sqrt{EI(t)}}, \quad (2.29)$$

$$\phi(s(x)) = [EI(x)]^{-1/4} \psi(x).$$

Since $\psi(x) = [EI(x)]^{1/4} \phi(s)$, the successive derivatives take the form

$$\psi'(x) = \frac{d}{dx} ([EI(x)]^{1/4} \phi(s)) = \frac{1}{4} [EI(x)]^{-3/4} (EI(x))' \phi(s) + [EI(x)]^{-1/4} \phi'(s), \quad (2.30)$$

$$\begin{aligned} \psi''(x) &= \frac{d}{dx} \psi'(x) = \frac{d}{dx} \left(\frac{1}{4} [EI(x)]^{-3/4} (EI(x))' \phi(s) + [EI(x)]^{-1/4} \phi'(s) \right) = \\ &= [EI(x)]^{-3/4} \phi''(s) + [EI(x)]^{-3/4} \left(\frac{(EI(x))''}{4} - \frac{3 ((EI(x))')^2}{16 EI(x)} \right) \phi(s), \end{aligned} \quad (2.31)$$

where $s = s(x)$.

After performing substitutions (2.29) – (2.31) and carrying out the full transformation, equation (2.26) takes the normalized form

$$\phi''(s) + \left(F + \frac{1}{4} (EI(x))'' - \frac{3 ((EI(x))')^2}{16 EI(x)} \right) \phi(s) = 0, \quad (2.32)$$

where $x = x(s)$, which can be written in the so-called Schrödinger form (so named due to its similarity to the time-independent Schrödinger equation studied in quantum mechanics)

$$\phi''(s) + (\lambda - V(s)) \phi(s) = 0, \quad (2.33)$$

where the *Liouville potential* is defined as

$$V(s) = \frac{3 ((EI(x))')^2}{16 EI(x)} - \frac{1}{4} (EI(x))'', \quad (2.34)$$

In Eqs. (2.32)–(2.34) the derivatives $EI'(x)$ and $EI''(x)$ are taken with respect to the original spatial coordinate x , whereas the second derivative ϕ'' is taken with respect to the Liouville coordinate s ; the potential $V(s)$ is obtained by evaluating these expressions at $x = x(s)$.

Relation (2.33) is the normalized form of the Sturm – Liouville equation, whose solutions – depending on the structure of the potential $V(s)$ – are expressed in terms of special functions such as Airy, Bessel, Hermite, or erf/erfi functions.

2.4. Significance of the Liouville transformation

The application of the Liouville transformation is of fundamental importance for the analysis of the stability of rods with variable bending stiffness $EI(x)$. Its main advantages are as follows:

- reduction of an arbitrary stiffness profile $EI(x)$ to a single eigenvalue equation; the transformation removes the weight function appearing in the Sturm – Liouville form and converts the equation into a normalized Schrödinger-type form, in which spectral analysis depends exclusively on the potential $V(s)$. This means that any rod with variable stiffness, described by an arbitrary function $EI(x)$, can be analyzed within a single unified mathematical model,
- unambiguous assignment of a special function to a given type of potential $V(s)$; the obtained equation $\phi''(s) + (\lambda - V(s))\phi(s) = 0$ is equivalent to equations known from the theory of special functions – thus, rods with different profiles can be classified by the type of differential equation, rather than by arbitrary approximations of the shape,
- analysis of the spectrum of the operator in the critical state is possible without the need to assume *a priori* buckling mode shapes; the transformation eliminates the first-derivative term, reducing the operator to a self-adjoint form – this allows eigenvalues to be studied using the theory of linear operators without the necessity of adopting traditional assumed buckling shapes $w(x)$ (e.g. trigonometric or polynomial), and avoids approximation errors, while the obtained solution has the status of an analytical or semi-analytical closed-form solution,
- the possibility of using symbolic CAS methods to compute eigenvalues λ , corresponding to the critical load F_{cr} ; the equation in the transformed variable s has a constant weight and a homogeneous form, which allows it to be solved directly using symbolic tools, employing known libraries of special functions and procedures for solving eigenvalue problems in Schrödinger form, which in turn translates into the fact that in many cases the solutions have a closed form, and even in more complex situations numerical computations are faster and more stable than for the original equation with variable coefficients in x .

The Liouville potential $V(s)$ plays a central role in the spectral formulation of the stability problem for rods with variable bending stiffness. In contrast to the original Euler–Bernoulli equation, where the variability of $EI(x)$ appears in multiplicative derivative terms, the potential $V(s)$ collects this information into a single effective term in the transformed Schrödinger-type equation. It should be emphasized that $V(s)$ is a mathematical quantity resulting from the Liouville transformation, and its interpretation depends on the normalization adopted. Therefore, it should not be directly identified with physical stiffening or weakening of the material. Instead, it determines the spectral properties of the operator and governs the distribution of eigenfunctions.

The Liouville transformation thus reveals in a direct way how the geometry of the rod $EI(x)$ influences the buckling mechanism and allows the stability problem to be analyzed in the language of linear operator theory, in exactly the same way as, in quantum mechanics, the energy spectrum of a particle in a force field is analyzed.

3. EXAMPLE OF MAPPING THE STIFFNESS PROFILE TO THE POTENTIAL AND SPECIAL FUNCTIONS

In this section, the operation of the Liouville transformation is illustrated using three representative bending stiffness profiles: constant, linearly varying, and Gaussian. In each case, the analysis is based on equation (2.26) and its Schrödinger-type form (2.32) – (2.33), in which the Liouville potential $V(s)$ uniquely corresponds to the selected second moment of area $I(x)$.

3.1. Constant profile

For a rolled rod with a constant second moment of area

$$I(x) = I_0 = \text{const}, EI(x) = EI_0, \quad (3.1)$$

the derivatives $(EI)'$ and $(EI)''$ vanish, hence

$$V(s) = 0. \quad (3.2)$$

The Schrödinger equation (2.33) then takes the form

$$\phi''(s) + \lambda \phi(s) = 0 \quad (3.3)$$

where $\lambda = F$. This is a classical equation with solutions in terms of trigonometric functions

$$\phi(s) = C_1 \sin(\sqrt{\lambda}s) + C_2 \cos(\sqrt{\lambda}s). \quad (3.4)$$

After imposing boundary conditions corresponding to pinned – pinned support (cf. Section 2.2), the eigenvalues and buckling modes known from Euler's theory are obtained, expressed in terms of sinusoidal functions.

3.2. Linear profile

Let us consider a simple linear profile (a conical rod in the sense of bending stiffness)

$$I(x) = I_0 (1 + \alpha x), \quad (3.5)$$

with $\alpha \in \mathbb{R}_+$, which gives

$$EI(x) = EI_0(1 + \alpha x). \quad (3.6)$$

The derivatives are

$$(EI(x))' = EI_0\alpha, \quad (EI(x))'' = 0, \quad (3.7)$$

and thus the potential becomes

$$V(x) = \frac{3EI_0\alpha^2}{16(1 + \alpha x)}. \quad (3.8)$$

The Liouville transformation introduces the change of variable

$$s(x) = \frac{1}{\sqrt{EI_0}} \int_0^x \frac{dt}{\sqrt{1 + \alpha t}} = \frac{2}{\alpha\sqrt{EI_0}} (\sqrt{1 + \alpha x} - 1), \quad (3.9)$$

which allows the dependence to be inverted to the form

$$1 + \alpha x = (1 + \alpha\sqrt{EI_0}s/2)^2, \quad (3.10)$$

and the potential to be written as

$$V(s) = \frac{3EI_0\alpha^2}{16(1 + \alpha\sqrt{EI_0}s/2)^2} = \frac{A}{(1 + \gamma s)^2}, \quad (3.11)$$

where $A = 3EI_0\alpha^2/16$ and $\gamma = \alpha\sqrt{EI_0}/2$.

After substitution, equation (2.33) takes the form

$$\phi''(s) + \left(\lambda - \frac{A}{(1+\gamma s)^2} \right) \phi(s) = 0. \quad (3.12)$$

Introducing a new variable

$$z = 1 + \gamma s \quad (3.13)$$

we obtain

$$\phi''(z) + \left(\Lambda - \frac{B}{z^2} \right) \phi(z) = 0 \quad (3.14)$$

where $\Lambda = \lambda/\gamma^2$ and $B = A/\gamma^2$.

It can be shown that, by means of the substitution $\phi(z) = z^{1/2}u(z)$, the equation can be reduced to the form

$$u''(z) + \frac{1}{z}u'(z) + \left(k^2 - \frac{\nu^2}{z^2} \right) u(z) = 0, \quad (3.15)$$

where $k^2 = \Lambda$ and $\nu^2 = B + 1/4$, which, after the substitution $kz = t$, becomes the classical Bessel equation (Żakowski et al. [45])

$$t^2 U'' + tU' + (t^2 - \nu^2)U = 0. \quad (3.16)$$

The solutions of equation (3.15) are expressed in terms of Bessel and Neumann functions of the appropriate order:

$$u(z) = C_1 J_\nu(\sqrt{\Lambda}z) + C_2 Y_\nu(\sqrt{\Lambda}z), \quad (3.17)$$

and thus

$$\phi(z) = z^{1/2} \left(C_1 J_\nu(\sqrt{\Lambda}z) + C_2 Y_\nu(\sqrt{\Lambda}z) \right), \quad (3.18)$$

here J_ν and Y_ν denote the Bessel functions of the first kind and the Bessel functions of the second kind (Neumann functions), respectively, of order ν (Żakowski et al. [41]).

After imposing boundary conditions corresponding to pinned – pinned support (cf. Section 2.2), the buckling modes are obtained.

3.3. Gaussian profile

Another example of a stiffness profile is the Gaussian-type profile, symmetric with respect to the center of the rod. Let us assume that the maximum stiffness occurs at the mid-span of the rod, i.e. at $x = L/2$ ($a > 0$):

$$I(x) = I_0 \exp(-a(x - L/2)^2), \quad (3.19)$$

and thus the bending stiffness is

$$EI(x) = EI_0 \exp(-a(x - L/2)^2). \quad (3.20)$$

The successive derivatives are

$$\begin{aligned} (EI(x))' &= EI(x) [-2a(x - L/2)], \\ (EI(x))'' &= EI(x) [4a^2(x - L/2)^2 - 2a], \end{aligned} \quad (3.21)$$

and hence

$$V(x) = EI(x) \left(\frac{a}{2} - \frac{a^2}{4} \left(x - \frac{L}{2} \right)^2 \right). \quad (3.22)$$

The Liouville transformation introduces the change of variable

$$s(x) = \frac{1}{\sqrt{EI_0}} \int_0^x \exp \left(\frac{a}{2} \left(t - \frac{L}{2} \right)^2 \right) dt. \quad (3.23)$$

Next, by introducing the substitution

$$u = \sqrt{\frac{a}{2}} \left(t - \frac{L}{2} \right), \quad (3.24)$$

we obtain

$$s(x) = \frac{1}{\sqrt{EI_0}} \sqrt{\frac{\pi}{2a}} \left(\operatorname{erfi} \sqrt{\frac{a}{2}} \left(x - \frac{L}{2} \right) - \operatorname{erfi} \left(-\sqrt{\frac{aL}{2}} \right) \right), \quad (3.25)$$

where $\operatorname{erfi}(\dots)$ denotes the imaginary error function (Marcinowski et al. [31]).

For convenience, a shifted variable can be introduced

$$\tilde{s}(x) = \sqrt{\frac{\pi}{2aEI_0}} \operatorname{erfi} \sqrt{\frac{a}{2}} \left(x - \frac{L}{2} \right), \quad (3.26)$$

which differs from $s(x)$ only by a constant; in local analysis (near the center of the rod) it can therefore be denoted simply by s . The function $x = x(s)$ is obtained by inversion of the erfi function. In order to obtain a simpler form of the eigenvalue equation and relate it to classical special functions, the behavior of the rod in the vicinity of the mid-span $x = L/2$ is analyzed. Let

$$y = x - L/2, \quad (3.27)$$

which yields

$$\begin{aligned} EI(x) &= EI_0 \exp(-ay^2), \\ V(x) &= EI_0 \exp(-ay^2) \left(\frac{a}{2} - \frac{a^2}{4} y^2 \right). \end{aligned} \quad (3.28)$$

Expanding near the center of the rod, for small y , using $\exp(-ay^2) \approx 1 - ay^2$, we obtain

$$V(x) = EI_0 \left(\frac{a}{2} - \frac{3a^2}{4} y^2 + \mathcal{O}(y^4) \right) \approx EI_0 \left(\frac{a}{2} - \frac{3a^2}{4} y^2 \right), \quad (3.29)$$

that is, locally

$$V(x) \approx V_0 - \omega y^2, \quad (3.30)$$

where $V_0 = aEI_0/2$, $\omega = 3a^2EI_0/4$.

On the other hand, from definition (3.23), in the surroundings of $x = L/2$, where $EI(x) \approx EI_0$

$$\frac{ds}{dx} = \frac{1}{\sqrt{EI(x)}} \approx \frac{1}{\sqrt{EI_0}} \Rightarrow s - s_0 \approx \frac{1}{\sqrt{EI_0}} (x - L/2) = \frac{y}{\sqrt{EI_0}}, \quad (3.31)$$

and thus

$$y \approx \sqrt{EI_0} (s - s_0). \quad (3.32)$$

Substituting (3.32) into (3.30), we obtain the local form of the potential in the Liouville variable

$$V(s) \approx V_0 - \beta (s - s_0)^2 \quad (3.33)$$

where $\beta = \omega EI_0 = 3a^2 E^2 I_0^2 / 4$.

Further, substituting (3.33) into equation (2.33), we obtain, in the local approximation

$$\phi''(s) + [\lambda - V_0 + \beta(s - s_0)^2]\phi(s) = 0. \quad (3.34)$$

Introducing the variable

$$\zeta = s - s_0 \quad (3.35)$$

and the scaling

$$\eta = \beta^{1/4} \zeta, \quad (3.36)$$

equation (3.34) takes the form

$$\phi''(\eta) + [\mu + \eta^2]\phi(\eta) = 0, \quad (3.37)$$

with $\mu = (\lambda - V_0)/\beta^{1/2}$. Equation (3.37) belongs to the class of Weber (parabolic cylinder) equations with an inverted parabolic potential. The Weber-type form obtained in the Gaussian case results from a local quadratic approximation of the Liouville potential and is used here solely for analytical classification of the considered equation; the buckling eigenfunctions remain real-valued and no extension into the complex plane is required. In the standard tabulated convention (Abramowitz et al. [46]), for the classical parabolic potential, solutions of the Weber equation are given in terms of Hermite polynomials for discrete parameter values. This means that the local shape of the buckling mode in the vicinity of the mid-span of the rod has the character of a Hermite-type function.

4. CLASSIFICATION OF SELECTED PROFILES $I(x)$, POTENTIALS $V(s)$ AND THE CORRESPONDING SPECIAL FUNCTIONS

The Liouville transformation reduces a broad class of stability equations of rods with variable bending stiffness $EI(x)$ to a normalized spectral form in which the Liouville potential $V(s)$ plays a central role. Different stiffness profiles $I(x)$ generate characteristic classes of potentials $V(s)$, which determine the associated differential equations whose solutions are expressed in terms of special functions such as Airy, Bessel, Hermite, and error-function-based representations. This establishes a systematic correspondence: geometric profile \rightarrow Liouville potential \rightarrow class of special functions, enabling the identification of both analytical structure and spectral properties of the buckling problem.

Table 1 summarizes the main stiffness profiles together with their associated Liouville potentials and corresponding special-function solutions.

The results are classified according to the level of mathematical representation, distinguishing between (i) exact analytical reductions, (ii) semi-analytical formulations, (iii) local approximations valid in the vicinity of selected points, and (iv) implicit transformations requiring numerical inversion of the Liouville coordinate. This distinction is essential for the correct interpretation of the analytical results presented in Table 1.

Table 1. Selected stiffness profiles and the corresponding forms of the potential and special functions

Stiffness profile $I(x)$	Type of analytical representation	Form of the potential $V(s)$ after transformation	Class of special functions describing $\phi(s)$	Engineering comment
constant profile $I(x) = I_0$	Exact	$V(s) = 0$	trigonometric functions	corresponds to the classical Euler rod
linear profile $I(x) = I_0(1 + \alpha x)$	Exact	$V(s) \sim c/s^2$	Bessel and Neumann functions	occurs in rods with a conical shape; analytical solutions available
quadratic profile $I(x) = I_0(1 + \beta x^2)$	Exact	$V(s) \sim \alpha + \beta s^2$	Hermite functions	most "regular" potential – linear spectrum; optimization
power-law profile $I(x) = I_0 x^m$	Exact	$V(s) \sim c/s^2$	Bessel and Hankel functions	classical family of buckling problems for power-law rods
exponential profile $I(x) = I_0 e^{kx}$	Exact	$V(s) = \text{const}$	trigonometric / hyperbolic functions	effectively homogeneous rod in the transformed variable
Gaussian profile $I(x) = I_0 e^{-a(x-L/2)^2}$	Local approximation	$V(s) \sim \alpha + \beta s^2$	Hermite functions	strong localization of buckling near the maximum stiffness
erfi-based profile $I(x) = I_0 e^{a \operatorname{erfi}(bx)}$	Implicit transformation	complex relation	Weber / Airy / Hermite functions	wide class of profiles for shape optimization
parabolic-exponential profile $I(x) = I_0(1 + \alpha x^2) e^{kx}$	Implicit transformation	highly nonlinear / implicit relation	parabolic / Whittaker functions	encountered in variable cross-section designs in aerospace engineering

5. CONCLUSIONS AND SUMMARY

The paper presents a unified analytical approach to the problem of buckling of compressed rods with variable bending stiffness, based on Sturm – Liouville operator theory and the Liouville transformation. It has been shown that the classical stability equation of the Euler – Bernoulli rod with non-uniform coefficients can be reduced to a normalized Schrödinger-type form, in which the key role is played by the potential uniquely determined by the profile of the cross-sectional moment of inertia.

Main conclusions of the study:

- A broad class of buckling problems of rods with variable bending stiffness can be reduced to a single Sturm – Liouville eigenvalue problem, in which the critical load appears as an eigenvalue of the stability operator. Differences between individual geometric profiles result not from different governing equations, but from different structures of the Liouville potential. This allows the previously scattered results in the literature to be treated as special cases of a coherent theory.
- The Liouville transformation reveals a direct relationship between the moment of inertia function, the structure of the stability operator spectrum, understood as a discrete set of eigenvalues $\{\lambda_i\}$. The

Liouville potential should therefore be understood as a transformed mathematical quantity that incorporates the influence of the stiffness distribution into the normalized spectral problem and governs the spectral properties of the corresponding operator.

- A classification of bending stiffness profiles through potentials and families of special functions describing eigenfunctions and eigenvalues has been proposed. This classification organizes existing analytical results and indicates in which cases exact or semi-analytical eigenvalue solutions can be obtained.
- Within the transformed analytical framework proposed in this study, there is no need for parameterization of arbitrary trial functions describing rod deflection (e.g. trigonometric or polynomial functions), as is typical of classical energy methods. The formulation is a direct consequence of applying a self-adjoint Sturm – Liouville operator, which increases mathematical rigor and reliability in determining eigenvalues corresponding to critical loads.
- The obtained normalized form of the stability equation creates a natural analytical framework for spectral classification of compressed rods, in which the design of the cross-sectional profile can be interpreted as a potential-shaping problem aimed at maximizing the first eigenvalue λ_1 (or a selected part of the spectrum), subject to mass or technological constraints. The presented approach is fundamental in nature and constitutes a starting point for further algorithmic research, in which the optimization problem can be formulated directly at the spectral level.
- Reducing the problem to a Schrödinger-type form provides a framework for the use of symbolic and semi-analytical tools available in CAS systems for the analysis of eigenvalues and eigenfunctions. In many cases, this leads to closed-form solutions or to numerical computations that are significantly more stable and efficient than those for the original fourth-order equation with variable coefficients.

In summary, the proposed analytical framework provides a coherent theoretical foundation for the analysis of compressed rods with variable bending stiffness. It combines classical structural mechanics with modern operator theory and special functions, organizing existing results and indicating new research directions, particularly in the analysis and design of non-prismatic structural elements, including full numerical examples, eigenvalue computations for selected stiffness profiles, and comparisons with FEM and Galerkin solutions.

6. NOMENCLATURE

- α – parameter describing the variability of the bending stiffness of the cross-section,
 E – Young's modulus,
 $EI(x)$ – bending stiffness of the rod,
 $I(x)$ – second moment of area of the cross-section,
 F – axial compressive force,
 F_{cr} – critical buckling load,
 $\phi(s)$ – eigenfunction in the normalized form of the equation,
 J_ν – Bessel function of the first kind of order ν ,
 λ – eigenvalue of the stability problem corresponding to the critical load F_{cr} ,
 L – length of the rod,
 p, q – coefficients of the Sturm – Liouville equation,
 r – weight function of the Sturm – Liouville equation,
 s – Liouville variable,
 $V(s)$ – Liouville potential,

- V_0 – value of the potential at its maximum,
 $w(x)$ – transverse deflection of the rod,
 x – coordinate along the rod axis,
 Y_ν – Bessel function of the second kind (Neumann function) of order ν .

REFERENCES

1. Euler, L 1744. Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutione problematica isoperitrici latesimo sensu accepti. Additamentum 1: De curvis elasticis. Lusanne and Geneva, *Apud Marcum-Michaelem, Bousquet et Socios*.
2. Lagrange, J L 1788. *Mecanique Analytique*. Courcier, Paris.
3. Timoshenko, S P 1966. History of strength of materials. (Historia wytrzymałości materiałów). *Wydawnictwo Arkady*, Warszawa.
4. Fairbairn, W 1858. On the resistance of tubes to collapse. *Philos. Trans. Roy. Soc. London*, vol. 148.
5. Bauschinger, J 1887. Mitteilungen aus dem mechanisch-technischen Laboratorium in München, Heft 15, pp. 11. München: *K. B. Technische Hochschule*.
6. Jasiński, F S 1893. Opyt razwitija teoriji prodolnogo izgiba. St. Petersburg, 1893.
7. Tetmajer, L 1896. Mitteilungen der Materialprüfungs - Anstalt am eidgenössischen Polytechnikum in Zürich, Heft VIII. Zürich: *Selbst-Verlag der eidgenössischen Festigkeits-Anstalt*.
8. Bresse, M 1859. Course de mecanique applique. P. 1. Paris, *Mallet-Bachelier, Imprimeur-Libraire du Bureau des Longitudes*.
9. Thompson, W 1863. Dyn. problems regarding elastic spheroidal shells and spheroids of incompressible liquid. *Philos. Trans. Roy. Soc. London*.
10. Lapunov, A M 1907. Problème général de la stabilité du mouvement. Traduit du russe (impr. en 1892 par la Soc. Math. Kharkow) par E. Dav. *Ann. Fan. Sci. Toulouse*, 2me serie, vol. 9.
11. Bryan, G H 1889. On the stability of elastic system. *Proc. Cambridge Philos. Soc.*, vol. 6.
12. Lagrange, J L 1770 – 1773. Sur la figure des colonnes. *Miscellanea Taurinensia*.
13. Clausen, T 1858. Über die Form architektonischer Säulen. *Bulletin physico-math. de l'Academie de St. Petersbourg*, t. IX.
14. Keller, JB and Tadjbakhsh, I 1960. The shape of the Strongest Column. *Arch. Rat. Mech. Anal.*, vol. 5, No. 4, pp. 275 – 285.
15. Keller, JB 1962. Strongest Column and Isoparametric Inequalities for Eigenvalues. *J. Appl. Mech.*, vol. 9, pp. 159 – 164.
16. Gajewski, A 1966. Calculation of Elastic Stability of Circular Plates with Variable Thickness by an Inverse Method. *Bull Acad. Polon. Sci., Ser. sci. techn.*, vol. 5, No. 14, pp. 303 – 312.
17. Gajewski, A 1975. Optimal strength-based shaping in the case of materials with physical nonlinearity (Optymalne kształtowanie wytrzymałościowe w przypadku materiałów o nieliniowości fizycznej). *Zesz. Nauk., P. Kr.*, nr 5, Kraków.
18. Filipow, AP and Griniew, WB 1975. Ob optymaln. oczertaniach strieżniej w zadaczach ustojczivosti. *Stroit, Mech. i Razcz. Sooruz.*, Nr 2, Moskwa, pp. 21 – 27.
19. Filipow, AP and Griniew WB 1977. Ob optymalnych strieżniach w zadaczach ustojczivosti pod diejstwijem raspredielennyh nagruzok. *Stroit, Mech. i Razcz. Sooruz.*, Nr 2, Moskwa, pp. 16 – 21.
20. Filipow, AP and Griniew WB 1979. Optimalizacja strieżniej po spektru sobstwiennyh zacznij. *Nauk. Dumka*, Kijew.

21. Krzyś, W, Życzkowski M 1966. A certain method of so-called parametric strength-based shaping. (Pewna metoda tzw. parametrycznego kształtowania wytrzyma.). *Rozpr. Inż.*, vol. 4, No. 11, pp. 643 – 666.
22. Błachut, J 1977. Optimal shaping of a compressed rod under large deflections using the dynamic programming method (Optymalne kształtowanie pręta ściskanego przy dużych ugięciach metodą programowania dynamicznego). *Mech. Teor. i Stos.*, No. 3, vol. 15, pp. 375 – 385.
23. Olhoff, N, Rasmussen S H 1977. On Single and Bimodal Optimum Buckling Loads of Clamped Columns. *Int. J. Solid Struct.*, vol. 13, pp. 605 – 614.
24. Marcinowski, J 2015. Maximum elastic buckling resistance of columns of constant volume. *XIV Symposium on Structural Stability*, Zakopane, 08–12.06.2015.
25. Timoshenko, S P, Gere, J M 1961. *Theory of Elastic Stability*. McGraw-Hill Book Company, New York.
26. Lee, B K, Ahn, D, Choi, J M, Lee, J K 2025. Buckling stability analysis of AFGM heavy columns with nonprismatic solid regular polygon cross-section and constant volume. *Science and Engineering of Composite Materials*, vol. 32, 20250064. DOI: 10.1515/secm-2025-0064
27. Love, A E H 1927. *A Treatise on the Mathematical Theory of Elasticity*. Cambridge University Press, Cambridge.
28. Konstantakopoulos, T G, Raftoyiannis, I G, Michaltsos, G T 2012. Stability of steel columns with non-uniform cross-sections. *The Open Construction and Building Technology Journal*, vol. 6, pp. 1–7. DOI: 10.2174/1874836801206010001
29. Lee, B K, Carr, A J, Lee, T E, Kim, I J 2006. Buckling loads of columns with constant volume. *Journal of Sound and Vibration*, vol. 294, No. 3–5, pp. 381–387. DOI: 10.1016/j.jsv.2005.11.004
30. Nikolić, R R, Djoković, J, Bujňák, J 2016. Buckling of variable-cross-sections columns in the braced and sway-plane frames. *Roczniki Inżynierii Budowlanej*, vol. 16 (2016).
31. Błażejowski, P, Klekiel, T, Kołodziej, S, Marcinowski, J, Sakharov, V 2023. Buckling resistance of metal columns with smoothly variable cross sections. *Archives of Civil Engineering*, vol. 69, No. 3, pp. 613–628. DOI: 10.24425/ace.2023.146101
32. Marcinowski, J, Sadowski, M 2020. Using the ERFI Function in the Problem of the Shape Optimization of the Compressed Rod. *International Journal of Applied Mechanics and Engineering*, vol. 25, No. 2, pp. 75–87. DOI: 10.2478/ijame-2020-0021.
33. Marcinowski, J, Sadowski, M 2021. Designing of Steel CHS Columns Showing Maximum Compression Resistance. *Civil and Environmental Engineering Reports*, vol. 31, No. 1, pp. 79–92. DOI: 10.2478/ceer-2021-0006.
34. Dokšanović, T, Radić, I, Biserčić, B 2023. Buckling Resistance of Tapered Steel Columns. *Applied Sciences*, vol. 13, No. 20, 11498. DOI: 10.3390/app132011498
35. Turkyilmazoglu, M 2024. Buckling phenomenon of vertical beam/column of variable density carrying a top mass. *Journal of Engineering Mathematics*, vol. 147, 4. DOI: 10.1007/s10665-024-10378-8.
36. Di Re, P 2025. Finite difference displacement integration for linear buckling analysis of non-uniform columns. *Mechanics Research Communications*, vol. 148, 104479. DOI: 10.1016/j.mechrescom.2025.104479
37. Love, A E H 1927. *A Treatise on the Mathematical Theory of Elasticity*. Cambridge University Press, Cambridge.
38. Romanów, F 1992. Structural stability (Stateczność konstrukcji). *Wydawnictwo Wyższej Szkoły Inżynierskiej w Zielonej Górze*.
39. Landau, L D, Lifshitz, E M 1970. *Theory of Elasticity*. Pergamon Press, Oxford.

40. Sadowski, M 2018. Spatial shaping of compressed rods with maximum buckling capacity. PhD thesis, University of Zielona Góra.
41. Volmir, A S 1967. Ustojchivost deformirujemych sistem. Moskwa: *Nauka*.
42. Coddington, EA, Levinson, N 1955. *Theory of Ordinary Differential Equations*. McGraw-Hill, New York.
43. Ince, EL 1927. *Ordinary Differential Equations*. London: *Longmans, Green and Co*. ISBN 978-1-108-76058-3
44. Titchmarsh, EC 1946. *Eigenfunction Expansions Associated with Second-Order Diff. Equations. Volume I*. Oxford: *Clarendon Press*. ISBN 978-0-19-853317-7.
45. Żakowski, W and Leksiński, W 1995. *Mathematics. Part IV. (Matematyka. Część IV)*. Wydanie dziewiąte. Warszawa: *Wydawnictwa Naukowo-Techniczne (WNT)*. ISBN 83-204-1896-8 (p. IV); ISBN 83-204-1893-3 (all).
46. Abramowitz, M and Stegun, IA 1964. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards, Applied Math. Series 55*.